Magnetic Waves Guided by a Linearly Tapered YIG Film

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***Abstract*—A quasi-optical treatment is given for the dispersion relation and the group delay time of a magnetic wave guided by a YIG film having a weak linear taper in its thickness in the propagation direction of the guided magnetic wave. This treatment has 1) confirmed the intuitive results in which the local value of the thickness is used for the tapered film, 2) indicated the frequency regions of validity of the intuitive results, and 3) revealed interesting features of the wavenormal and ray directions inside a YIG film.**

I. Introduction

PLANAR LAYERS of yttrium iron garnet (YIG) are employed in delay lines using guided magnetic waves for microwave frequencies [1], [2]. It is usual to resort to the wave theory for obtaining the dispersion relation and the group delay time. However, quasi-optic considerations can also be used for determining the dispersion relation [3] and the group delay time [4], [5]. Recently [6] we have given such a quasi-optic treatment of the dispersion relation and the group delay time of the magnetic waves guided by a planar film of YIG for the case in which the film is magnetized normal to the surfaces or parallel to the surfaces and to the direction of propagation of the guided wave. Since the quasi-optic procedure employs local relations, such a technique can be used for the treatment of film geometries not readily amenable to a wave-theoretical treatment. In this paper, we give a treatment of the dispersion relation and the group delay time of the magnetic waves guided by a YIG film having a weak linear taper in its thickness in the propagation direction of the guided magnetic wave. We have considered the magnetization direction normal to the midplane of the film and parallel to the midplane of the film and to the propagation direction of the guided magnetic wave. Our treatment has confirmed the intuitive results in which the thickness of a planar film is replaced by the local value of the thickness for a tapered film. In addition our treatment has shown the frequency regions of validity of the intuitive results and has revealed interesting features of the wavenormal and ray directions inside a YIG film.

II. Normal Magnetization

A nearly planar slab of YIG ($\mu_0\mu_r, \epsilon_0\epsilon_r$) situated in free space (μ_0, ϵ_0) has a weak linear taper in the x-direction of its thickness. A Cartesian coordinate system is chosen

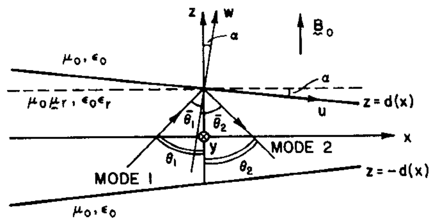

Fig. 1. Geometry of the tapered YIG film in air for the normal magnetization. $d(x) = d - x\tan\alpha$.

such that the plane $z=0$ coincides with the midplane of the film as shown in Fig. 1. The geometry of the film has no variation in the y-direction. The film occupies the region $-\infty < x, y < \infty$ and $-d(x) < z < d(x)$ where $d(x) = d - x\tan\alpha$ and α is the angle of the taper. The average thickness of the film is $2d$ which is the same as the actual thickness at $x=0$. The film is uniformly magnetized in the z-direction which is nearly normal to the surfaces of the film. All the field quantities are assumed to be independent of y and have the harmonic time dependence of the form $\exp(-i\omega t)$. For the magnetic waves, inside the film, the magnetic scalar potential satisfies the differential equation

$$\left(\mu_1 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x, z) = 0. \tag{1}$$

The magnetic field $\boldsymbol{H}(x, z)$ and the magnetic flux density $\boldsymbol{B}(x, z)$ are related to $\psi(x, z)$ as follows:

$$H_x(x, z) = \frac{\partial}{\partial x}\psi(x, z) \tag{2a}$$

$$H_y(x, z) = 0 \tag{2b}$$

$$H_z(x, z) = \frac{\partial}{\partial z}\psi(x, z) \tag{2c}$$

$$B_x(x, z) = \mu_0\mu_1 \frac{\partial}{\partial x}\psi(x, z) \tag{3a}$$

$$B_y(x, z) = -i\mu_0\mu_2 \frac{\partial}{\partial x}\psi(x, z) \tag{3b}$$

$$B_z(x, z) = \mu_0 \frac{\partial}{\partial z}\psi(x, z). \tag{3c}$$

We need only the component μ_1 of $\boldsymbol{\mu}_r$ and it is given by

$$\mu_1 = \left[\omega^2 - \omega_H(\omega_H + \omega_M)\right] / \left(\omega^2 - \omega_H^2\right) \tag{4}$$

where ω_H is the gyromagnetic angular frequency and ω_M is the angular frequency corresponding to the saturation magnetization. The fields in free space are also governed

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by (1), (2), and (3) with $\mu_1=1$ and $\mu_2=0$.

For an unbounded YIG, assuming a solution of the form $\psi(x,z)\sim\exp(i\boldsymbol{\beta}\cdot\boldsymbol{r})$ with $\boldsymbol{\beta}=\hat{x}\beta_x+\hat{z}\beta_z$, we obtain the dispersion relation as $\beta_z=(-\mu_1)^{1/2}\beta_x$ which together with (4) shows that a homogeneous plane wave corresponding to real β_x and β_z is possible only for $\mu_1<0$ or equivalently for $\omega_H<\omega<[\omega_H(\omega_H+\omega_M)]^{1/2}$. We assume that ω is restricted to this range leading to the possibility of homogeneous plane waves in the YIG. From the dispersion relation $\beta_z=(-\mu_1)^{1/2}\beta_x$ in an unbounded YIG, we can show that the group velocity $\boldsymbol{v}_g$ is perpendicular to the wave vector $\boldsymbol{\beta}$, that is, $\boldsymbol{v}_g\cdot\boldsymbol{\beta}=0$.

A plane magnetic wave is incident on the upper interface $z=d(x)$ between the YIG and the free space from the side of the YIG as shown in Fig. 1. There is a reflected magnetic wave in the YIG and a transmitted magnetic field in the free space. This reflection phenomenon can be treated conveniently in a rotated coordinate system (u,y,w) as defined by

$$u=x\cos\alpha-(z-d)\sin\alpha \tag{5a}$$

$$w=x\sin\alpha+(z-d)\cos\alpha \tag{5b}$$

where the u-axis is parallel to the upper surface of the film. In the rotated coordinate system, (1) becomes

$$\left[(\mu_1\cos^2\alpha+\sin^2\alpha)\frac{\partial^2}{\partial u^2}-2(1-\mu_1)\sin\alpha\cos\alpha\frac{\partial^2}{\partial u\partial w}+(\mu_1\sin^2\alpha+\cos^2\alpha)\frac{\partial^2}{\partial w^2}\right]\psi(u,w)=0. \tag{6}$$

The magnetic scalar potential in the YIG is of the form

$$\psi(u,w)=\left[A_1\exp(i\beta_{w1}w)+A_2\exp(i\beta_{w2}w)\right]\exp(i\beta_u u) \tag{7}$$

where β_u is assumed to be positive and β_{w1} and β_{w2} can be obtained in terms of β_u from (6) as follows:

$$\beta_{w1}=\frac{\left[(-\mu_1)^{1/2}+(1-\mu_1)\sin\alpha\cos\alpha\right]}{\left[\mu_1\sin^2\alpha+\cos^2\alpha\right]}\beta_u \tag{8a}$$

$$\beta_{w2}=\frac{\left[-(-\mu_1)^{1/2}+(1-\mu_1)\sin\alpha\cos\alpha\right]}{\left[\mu_1\sin^2\alpha+\cos^2\alpha\right]}\beta_u. \tag{8b}$$

From (6) with $\mu_1=1$, we find the magnetic scalar potential in the free space to be of the form

$$\psi(u,w)=A_3\exp(-\beta_u w)\exp(i\beta_u u). \tag{9}$$

The tangential magnetic field $H_u(u,w)$ and the normal magnetic flux density $B_w(u,w)$ can be found from (2), (3), and (5). Applying the boundary conditions that $H_u(u,w)$ and $B_w(u,w)$ are continuous at the interface $w=0$ yields

$$1/R=A_2/A_1=\exp(i2\phi) \tag{10}$$

where

$$\phi=\tan^{-1}\left[(-\mu_1)^{1/2}\right]+\pi/2. \tag{11}$$

A_2/A_1 is designated as the phase reflection coefficient. It is important to note that the phase reflection coefficient, being independent of α, is the same for all angles of taper of the film. Also, since ϕ is not a function of β_u, β_{w1}, and β_{w2}, it follows that the phase reflection coefficient is independent of the angles of incidence and reflection of the magnetic wave in the YIG film.

The plane waves having the wavenumbers β_{w1} and β_{w2} in the w-direction are designated as modes 1 and 2, respectively, in an unbounded YIG. In the w-direction, it can be shown from (8) that the group velocity of the mode 1 is negative and that of the mode 2 is positive, i.e.,

$$v_{gw1}<0 \qquad v_{gw2}>0. \tag{12}$$

Therefore, the mode 2 represents a wave whose energy travels towards the interface or whose associated ray is directed towards the interface. Hence, the mode 2 represents the incident ray. Similarly, the mode 1 represents the reflected ray since the energy associated with this wave travels away from the interface. Consequently, the ray reflection coefficient $R=A_1/A_2=\exp(-i2\phi)$ as indicated in (10).

Let $\bar{\theta}_1$ be the angle made by the wavenormal of the mode 1 with the w-axis and $\bar{\theta}_2$ be the angle made by the negative of the wavenormal of the mode 2 with the w-axis, as shown in Fig. 1. Also, let θ_1 and θ_2 be the angles with respect to the z-axis corresponding to $\bar{\theta}_1$ and $\bar{\theta}_2$, respectively, as also indicated in Fig. 1. Therefore

$$\tan\bar{\theta}_1=\beta_u/\beta_{w1} \tag{13a}$$

$$\tan\bar{\theta}_2=-\beta_u/\beta_{w2} \tag{13b}$$

$$\theta_1=\bar{\theta}_1+\alpha \tag{14a}$$

$$\theta_2=\bar{\theta}_2-\alpha. \tag{14b}$$

Using (8), (13), and (14) we find the important result that

$$\theta_1=\theta_2=\tan^{-1}\left[(-\mu_1)^{-1/2}\right]. \tag{15}$$

The wavenormal of the mode 1 and the negative of the wavenormal of the mode 2 make equal angles with the direction (z) of magnetization.

The reflection of magnetic waves at the lower interface $z=-d(x)$ separating the YIG($z>-d(x)$) from the free space ($z<-d(x)$) can also be treated in a similar manner for the wave incident from the side of the YIG. As before, we use a rotated coordinate system (u,y,w) such that the u-axis is parallel to the lower interface of the film. Therefore, u and w are defined as in (5a), (5b) with α and d replaced by $-\alpha$ and $-d$, respectively. Consequently, for both the interfaces, we obtain the same value for the phase as well as the ray reflection coefficients. Also, at the lower interface, the wavenormals of the modes 1 and 2 make equal angles with the direction (z) of magnetization.

Since the film is homogeneous, the wavenormal directions inside the film are straight lines. In view of (15) the wavenormal directions of the mode 1 successively and totally internally reflected from the opposite surfaces of the film are parallel even though the two reflecting interfaces are not parallel. A similar result holds good for the mode 2 also. Since the rays are normal to the corresponding wavenormals, the zig-zagging ray directions obtained from successive reflections at the two interfaces are such

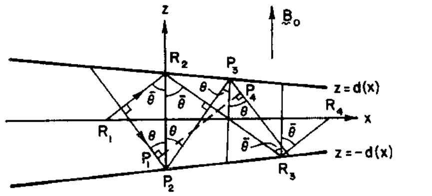

Fig. 2. Zig-zag model for the wavenormal directions and the associated zig-zagging ray directions for the normal magnetization. $\bar{\theta}=\pi/2-\theta$.

that all the rays directed towards or away from a particular interface are parallel. Thus, the directions of the zig-zagging wavenormals and the rays inside a YIG film with parallel interfaces remain unchanged when the two interfaces are slanted with respect to the midplane of the film so as to obtain a linear taper in the thickness.

The ray of the magnetic wave on reflection at a plane interface between the YIG and the free space undergoes, in general, a lateral displacement [7]–[9] in the x-direction as given by $2x_s=2\partial\phi/\partial\beta_x$ and an associated time delay [4] as given by $2t_s=-2\partial\phi/\partial\omega$. In general, the phase term ϕ is a function of the independent variables ω and β_x. As pointed out previously [6], since ϕ is independent of β_x, the ray of the magnetic wave does not undergo a Goos–Hänchen lateral displacement. However, there is a non-vanishing group delay on reflection at a plane interface between the YIG and the free space.

The dispersion relation of the magnetic wave guided along the YIG film can be deduced using the zig-zag model for the propagation of the phase fronts of a homogeneous plane wave inside the film [3]. The zig-zagging wavenormal directions of the successively reflected homogeneous plane wave inside the film are shown by the solid line $P_1P_2P_3P_4$ in Fig. 2. The angle made by the wavenormals of the modes 1 and 2 with the direction (z) of magnetization is denoted by θ. The dashed line P_1P_4 represents a phase front, namely, that is perpendicular to the wavenormal of the mode 2. We assume that the thickness of the film varies sufficiently slowly in the x-direction that in the distance between P_1 and P_4, the film thickness is essentially a constant. Therefore, in obtaining the closure condition on the phase between P_1 and P_4, we can assume the two interfaces of the film to be locally parallel. Since the closure condition is a local relation, it can be deduced for a tapered film essentially in the same manner as for a planar film with parallel interfaces [6]. The result is

$$4(-\mu_1)^{1/2}d(x)\beta_x+4\phi=2\pi(n+1), \qquad n=1,2,3,\cdots,\text{etc.} \tag{16}$$

where the right-hand side is chosen such that $n=1$ corresponds to the smallest possible positive value for β_x. The closure condition (16) is the dispersion relation of the guided magnetic wave. The group velocity V_{gx} of the guided magnetic wave can be determined from (16) as

$$V_{gx}=(-\mu_1)^{1/2}d(x)\left[(-\mu_1)^{1/2}d(x)/v_{gx}-\partial\phi/\partial\omega\right]^{-1} \tag{17}$$

where v_{gx} is the x-component of the group velocity in the unbounded YIG.

The distance between P_1 and P_4 is approximately equal to $4d\tan\theta=4d(-\mu_1)^{-1/2}$. The fractional change in the thickness of the YIG film from P_1 to P_4 is obtained as $4(-\mu_1)^{-1/2}\tan\alpha$. This fractional change in the film thickness has to be very small compared to unity in order for us to be able to treat the closure condition as a local relation. Consequently, (16) and (17) are valid except near $\mu_1=0$, that is except for frequencies very close to the resonant frequency $\omega=[\omega_H(\omega_H+\omega_M)]^{1/2}$.

Since $\boldsymbol{v}_g\cdot\boldsymbol{\beta}=0$, the rays inside the film are perpendicular to the corresponding wavenormal directions. Therefore, there are zig-zagging ray directions inside the film associated with the zig-zagging wavenormal directions. It has been pointed out previously that, on reflection at an interface between the YIG and the free space, a ray does not undergo a lateral shift. In Fig. 2 $R_1R_2R_3R_4$ is a part of the zig-zagging ray directions. These ray directions correspond to $P_1P_2P_3P_4$ which is a part of the zig-zagging wavenormal directions. The angle made by the ray directions associated with the modes 1 and 2 with the direction (z) of magnetization is denoted by $\bar{\theta}$ where $\bar{\theta}=\pi/2-\theta$. R_1R_4 is on the midplane $z=0$ of the YIG film. We assume that the thickness of the film varies sufficiently slowly in the x-direction that in the distance between R_1 and R_4, the film thickness is essentially a constant. Therefore, in obtaining the group velocity by the ray method, we can assume that the two interfaces of the film are locally parallel. Since the ray method involves only local relations, the group velocity for a tapered film by the ray method can be deduced with the help of a procedure similar to that employed for a planar film with parallel interfaces. The result is the same as that given by (17). Hence, there is internal consistency since the ray method gives the same result for the group velocity of the guided magnetic wave on a tapered film as that obtained by the wave method and the dispersion relation.

The distance between R_1 and R_4 is equal to $4d\tan\bar{\theta}=4d(-\mu_1)^{1/2}$. The fractional change in the thickness of the YIG film from R_1 to R_4 is obtained as $4(-\mu_1)^{1/2}\tan\alpha$. This fractional change in the film thickness has to be very small compared to unity in order to validate the local relations used in the ray method of obtaining the group velocity. Consequently, for a weakly tapered film, the ray method of determining the group velocity is valid except for frequencies very close to the cutoff frequency $\omega=\omega_H$ where μ_1 tends to infinity. It should be noted that near the cutoff frequency even the validity of the magnetic wave approximation used in the present analysis breaks down. Thus, the quasi-optic method of deducing the dispersion relation and the group velocity of the magnetic wave guided by a weakly tapered YIG film is valid except in the neighborhood of the cutoff and resonant frequencies. We shall now proceed to give a brief treatment for the magnetization parallel to the propagation direction of the guided magnetic wave and nearly parallel to the surfaces of the film.

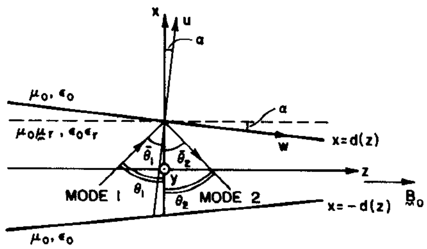

Fig. 3. Geometry of the tapered YIG film in air for the parallel magnetization. $d(z)=d-z\tan\alpha$.

III. Parallel Magnetization

It is convenient to keep the magnetization in the z-direction. For the nearly planar slab of YIG situated in free space, the weak linear taper in the thickness now occurs in the z-direction. We now choose a Cartesian coordinate system such that the plane $x=0$ coincides with the midplane of the film as shown in Fig. 3. The geometry of the film has no variation in the y-direction. The YIG film now occupies the region $-d(z)<x<d(z)$ and $-\infty<y, z<\infty$ where $d(z)=d-z\tan\alpha$. The film is uniformly magnetized in the z-direction which is nearly parallel to the surfaces of the film. As before, all the field quantities are assumed to be independent of y. For the magnetic waves, the governing equations (1)–(3) and the plane wave characteristics as given in Section II are unaltered.

A plane magnetic wave is incident on the upper interface $x=d(z)$ between the YIG and the free space from the side of the YIG as shown in Fig. 3. There is a reflected magnetic wave in the YIG and a transmitted magnetic field in the free space. To analyze this reflection phenomenon, we use a rotated coordinate system (u, y, w) as defined by

$$u=(x-d)\cos\alpha+z\sin\alpha \tag{18a}$$

$$w=-(x-d)\sin\alpha+z\cos\alpha \tag{18b}$$

where the w-axis is parallel to the upper surface of the film. In the new coordinate system, (1) becomes

$$\left[(\mu_1\cos^2\alpha+\sin^2\alpha)\frac{\partial^2}{\partial u^2}+2(1-\mu_1)\sin\alpha\cos\alpha\frac{\partial^2}{\partial u\,\partial w}+(\mu_1\sin^2\alpha+\cos^2\alpha)\frac{\partial^2}{\partial w^2}\right]\psi(u,w)=0. \tag{19}$$

We take the magnetic scalar potential in the YIG to be of the form

$$\psi(u,w)=\left[A_1\exp(i\beta_{u1}u)+A_2\exp(i\beta_{u2}u)\right]\exp(i\beta_w w) \tag{20}$$

where β_w is assumed to be positive, and β_{u1} and β_{u2} can be expressed in terms of β_w with the help of (19) as follows:

$$\beta_{u1}=\frac{\left[-(-\mu_1)^{1/2}-(1-\mu_1)\cos\alpha\sin\alpha\right]}{\left[\mu_1\cos^2\alpha+\sin^2\alpha\right]}\beta_w \tag{21a}$$

$$\beta_{u2}=\frac{\left[(-\mu_1)^{1/2}-(1-\mu_1)\cos\alpha\sin\alpha\right]}{\left[\mu_1\cos^2\alpha+\sin^2\alpha\right]}\beta_w. \tag{21b}$$

Using (19) with $\mu_1=1$, we express the transmitted magnetic scalar potential in the free space as

$$\psi(u,w)=A_3\exp(-\beta_w u)\exp(i\beta_w w). \tag{22}$$

The tangential magnetic field $H_w(u,w)$ and the normal magnetic flux density $B_u(u,w)$ can be determined from (2), (3), and (18). Applying the boundary conditions that $H_w(u,w)$ and $B_u(u,w)$ are continuous at the interface $u=0$ yields

$$R=A_2/A_1=\exp(-i2\phi) \tag{23}$$

where ϕ is the same as that given in (11). The phase reflection coefficient A_2/A_1 has the same important features as those for the normal magnetization.

The plane waves having the wavenumbers β_{u1} and β_{u2} in the u-direction are designated as modes 1 and 2, respectively, in an unbounded YIG. In the u-direction, it can be shown from (21) that the group velocity of the mode 1 is positive and that of the mode 2 is negative, i.e.,

$$v_{gu1}>0 \qquad v_{gu2}<0. \tag{24}$$

Hence, the modes 1 and 2 represent the incident and reflected rays, respectively. Consequently, the phase reflection coefficient A_2/A_1 and the ray reflection coefficient R are the same as indicated in (23).

Let $\bar{\theta}_1$ be the angle made by the wavenormal of the mode 1 with the u-axis and $\bar{\theta}_2$ be the angle made by the negative of the wavenormal of the mode 2 with the u-axis, as shown in Fig. 3. Also, θ_1 and θ_2 be the angles with respect to the x-axis corresponding to $\bar{\theta}_1$ and $\bar{\theta}_2$, respectively, as also indicated in Fig. 3. Hence,

$$\tan\bar{\theta}_1=\beta_w/\beta_{u1} \tag{25a}$$

$$\tan\bar{\theta}_2=-\beta_w/\beta_{u2} \tag{25b}$$

$$\theta_1=\bar{\theta}_1+\alpha \tag{26a}$$

$$\theta_2=\bar{\theta}_2-\alpha. \tag{26b}$$

Using (21), (25), and (26) we obtain the important result that

$$\theta_1=\theta_2=\tan^{-1}\left[(-\mu_1)^{1/2}\right]. \tag{27}$$

The wavenormal of the mode 1 and the negative of the wavenormal of the mode 2 make equal angles with the x-direction, which is perpendicular to the magnetization direction.

The reflection of magnetic waves at the lower interface $x=-d(z)$ separating the YIG $(x>-d(z))$ from the free space $(x<-d(z))$ can also be investigated in a similar manner for the wave incident from the side of the YIG. For both the interfaces, we obtain the same value for the phase as well as the ray reflection coefficients. At the lower interface also, the wavenormal directions of the modes 1 and 2 make equal angles with the x-direction, which is normal to the magnetization direction.

As with the normal magnetization, for the parallel magnetization also, the wavenormal directions of the mode 1 as well as those of the mode 2 successively and totally internally reflected from the opposite surfaces of the film

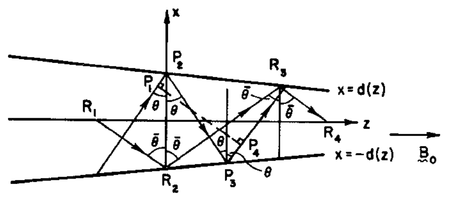

Fig. 4. Zig-zag model for the wavenormal directions and the associated zig-zagging ray directions for the parallel magnetization. $\bar{\theta}=\pi/2-\theta$.

are parallel even though the two reflecting interfaces are not parallel. Similar results are obtained for the zig-zagging ray directions associated with the modes 1 and 2. It follows therefore that even for the parallel magnetization, the directions of the zig-zagging wavenormals and the rays inside a YIG film with parallel interfaces remain unchanged when the two interfaces are slanted with respect to the midplane of the film so as to obtain a linear taper in the thickness. Also, for the parallel magnetization, the ray reflection coefficient is the same as that for the normal magnetization; hence, the ray on reflection at a plane interface between the YIG and the free space, undergoes no lateral displacement but has a time delay identical to that given for the normal magnetization, that is $2t_s=-2\partial\phi/\partial\omega$.

The zig-zagging wavenormal directions of the successively reflected homogeneous plane wave inside the film are shown by the solid line $P_1P_2P_3P_4$ in Fig. 4. The angle made by the wavenormals of the modes 1 and 2 with the x-direction is denoted by θ. The dashed line P_1P_4 represents a phase front, namely, that is perpendicular to the wavenormal of the mode 1. The film thickness is assumed to vary sufficiently slowly in the z-direction that in the distance between P_1 and P_4, the film thickness is essentially a constant. Therefore, for determining the closure condition on the phase between P_1 and P_4, the two interfaces may be considered to be locally parallel. Hence, the closure condition for a tapered film can be deduced essentially in the same manner as for a planar film with parallel interfaces [6]. The result is

$$4(-\mu_1)^{-1/2}d(z)\beta_z-4\phi=2\pi(n-2), \qquad n=1,2,3,\cdots,\text{etc.} \tag{28}$$

where the right-hand side is chosen such that $n=1$ corresponds to the smallest possible positive value for β_z. The closure condition (28) is the dispersion relation of the guided magnetic wave. The group velocity V_{gz} of the guided magnetic wave can be derived from (28) as

$$-V_{gz}=(-\mu_1)^{-1/2}d(z)\left[(-\mu_1)^{-1/2}d(z)/(-v_{gz})-\partial\phi/\partial\omega\right]^{-1} \tag{29}$$

where v_{gz} is the z-component of the group velocity in the unbounded YIG.

The distance between P_1 and P_4 is approximately equal to $4d\tan\theta=4d(-\mu_1)^{1/2}$. The fractional change in the thickness of the YIG film from P_1 to P_4 is obtained as $4(-\mu_1)^{1/2}\tan\alpha$. This fractional change has to be very small compared to unity to justify the treatment of the closure condition as a local relation. Therefore, (28) and (29) are valid except when μ_1 becomes very large, that is, except for frequencies very close to the resonant frequency $\omega=\omega_H$.

The rays are perpendicular to the corresponding wavenormals and do not undergo lateral shifts on reflection. There are zig-zagging rays associated with the zig-zagging wavenormals. In Fig. 4 $R_1R_2R_3R_4$ is a part of the zig-zagging ray directions. These ray directions correspond to $P_1P_2P_3P_4$ which is a part of the zig-zagging wavenormal directions. The angle made by the ray directions associated with the modes 1 and 2 with the x-direction is denoted by $\bar{\theta}$ where $\bar{\theta}=\pi/2-\theta$. R_1R_4 is on the midplane $x=0$ of the YIG film. We assume that the slow variation of the film thickness in the z-direction is such that in the distance between R_1 and R_4, the film thickness is essentially a constant. Therefore, for deducing the group velocity by the ray method, the two interfaces of the film may be approximated to be locally parallel. Hence, the group velocity for a tapered film by the ray method can be derived with the help of a procedure similar to that employed for a planar film with parallel interfaces. The result is the same as that given by (29). As with the normal magnetization, for the parallel magnetization also, there is internal consistency since the ray method gives the same result for the group velocity of the guided magnetic wave on a tapered film as that determined by the wave method and the dispersion relation.

The distance between R_1 and R_4 is equal to $4d\tan\bar{\theta}=4d(-\mu_1)^{-1/2}$. The fractional change in the thickness of the film from R_1 to R_4 is $4d(-\mu_1)^{-1/2}\tan\alpha$. This fractional change has to be very small compared to unity in order to validate the ray method of determining the group velocity. Hence, for a weakly tapered film, the ray method of finding the group velocity is valid except for frequencies very close to the cutoff frequency $\omega=[\omega_H(\omega_H+\omega_M)]^{1/2}$ where $\mu_1=0$. Near the cutoff frequency, the magnetic wave approximation used in the present analysis also breaks down. Thus, as with the normal magnetization, for the parallel magnetization also, the quasi-optic method of deducing the dispersion relation and the group velocity of the magnetic wave guided by a weakly tapered YIG film is valid except in the neighborhood of the cutoff and resonant frequencies.

IV. Concluding Remarks

There are two magnetic wave modes in an unbounded YIG with a uniform magnetization. The wavenormals of these two modes make equal angles with either the direction of magnetization or the direction which is perpendicular to the magnetization direction. In addition to the saturation magnetization and the applied magnetic flux, the angle of inclination depends only on the wave frequency. In a bounded YIG such as in a thin film, where the homogeneous plane wave modes are multiply and totally internally reflected from the two interfaces of the film, there are an infinite number of guided modes having different wavenumbers. For all these guided modes, the aforementioned inclination angle remains the same. This feature of the wavenormals of the homogeneous plane wave modes is unaffected by the geometry of the film. In particular, the angle of inclination is the same for a film with parallel interfaces having a constant separation distance between them and for a film with nonparallel interfaces having a separation distance which varies in the propagation direction. The rays are perpendicular to the corresponding wavenormals. The rays associated with the plane wave modes also make equal angles with either the direction of magnetization or the direction which is perpendicular to the magnetization direction. As in the case of the wavenormals, for the rays also, the angle of inclination depends only on the wave frequency and does not depend on the order of the guided mode or the geometry of the film.

Inside a film there are zig-zagging wavenormal directions caused by the successive and total internal reflection of the plane wave modes at the two interfaces. The wavenormals of each of the two plane wave modes are always parallel. This parallelism is obtained even if the two interfaces are slanted significantly. For the application of the closure condition on the phase, we require the fractional change in the thickness of the film in one unit of the zig-zagging wavenormal directions to be very small compared to unity. Such a small fractional change in the thickness is possible only for weakly slanted interfaces and that too for frequencies not close to the guided wave resonant frequency. The application of the closure condition yields the dispersion relation of the guided-wave modes. Similarly, associated with the zig-zagging wavenormals, there are zig-zagging rays. The rays of each of the two plane wave modes are always parallel. Although this parallelism is obtained even if the two interfaces are slanted significantly, in order to justify the local relations used in the derivation of the group velocity by the ray method, we require the fractional change in the thickness of the film in one unit of the zig-zagging ray directions to be very small compared to unity. This small fractional change in the film thickness can be obtained only if the two interfaces are slanted slightly and that too for frequencies not close to the guided-wave cutoff frequency. The ray method of determining the group velocity yields a result identical to that obtained from the dispersion relation.

In this paper we have given the details for the various results for the magnetization direction 1) normal to the midplane of the film, and 2) parallel to the propagation direction of the guided magnetic wave and the midplane of the film. However, one of us has obtained similar results for a more general case in which the magnetization direction lies in a plane normal to the midplane of the film but containing the propagation direction.

For a slightly tapered film, intuitive considerations lead us to replace the actual thickness in the dispersion relation and the group velocity of the magnetic wave guided by a planar film with parallel interfaces by the local value of the thickness for a slightly tapered film with nonparallel interfaces. Our quasi-optic treatment of the magnetic waves guided by a weakly tapered YIG film has confirmed the intuitive results and has also clarified the frequency ranges of validity of the intuitive results. In addition, our treatment has revealed the interesting features of the wavenormal and the ray directions inside a YIG film with a uniform magnetization.

Our treatment is valid for any kind of taper of the film so long as the taper is sufficiently weak. When an uniform external magnetic flux is applied, the applied magnetic flux inside the YIG film is not necessarily uniform. Therefore, it is necessary to assess also the effect of the weak inhomogeneity in the applied magnetic flux inside the YIG film. Since the ray directions of the two plane wave modes depend on the direction of the applied magnetic flux inside the film, it is considerably more difficult to include the effect of the weak inhomogeneity in the applied magnetic flux inside the film.

For a planar film of length L, thickness $2d$, and parallel (p) interfaces, let the group delay time of the guided magnetic wave be denoted by T_p. Similarly, let T_t be the corresponding group delay time for a linearly tapered (t) film of the same length L, the same average thickness $2d$ and nonparallel interfaces. From the results obtained in this investigation, we find that $T_t/T_p = 1 + L^2 \tan^2 \alpha / 12d^2$ where α is the small tapering angle. Thus, a weak linear taper in the film thickness tends to increase the group delay time and this increment factor is the same for all the frequencies and for all orders of the guided modes.

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